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Variational extensions of classical mechanics: stochastic and quantum dynamics

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Abstract
Variational extensions of the classical mechanics of the particles are considered, on the basis of a re-formulation of Hamilton's principle aimed at a more general approach to the analysis of evolution phenomena. Some generalized forms of stochastic and, consequently, quantum mechanics are then obtained.

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1. Introduction

In this paper, we show how some extensions or generalizations of the deterministic classical dynamics can be obtained in a 'natural' way, if we start from a more flexible and appropriate formulation of Hamilton's action principle. As a matter of fact, this re-formulation of the classical action principle can provide a more general and powerful tool for the investigation of the laws of evolution phenomena. The extensions which we will consider are particularly addressed to stochastic and quantum mechanics.

In order to delineate the basic features of our approach, we will consider, for the sake of simplicity, the motion of a particle of mass m, in a field of forces described by a potential $U(x)$. We assume that the configuration space is $\mathbb{R}^n$ (so $x \in \mathbb{R}^n$) and that $U(x)$ is a smooth function.

The basic ingredients of classical mechanics are the trajectories $x(t)$ of the particle. These are determined variationally by Hamilton's principle, through the Lagrangian

$$L(x(t), \dot{x}(t)) = \tfrac{1}{2}m(\dot{x}(t))^2 - U(x(t)) \qquad (x(t) \in \mathbb{R}^n). \tag{1}$$

As a matter of fact, classical mechanics shows that the motion of the particle can also be described in terms of 'wave fronts'. This is achieved through the Hamilton–Jakobi PDE

$$\partial_t S(x,t) + \frac{1}{2m}(\nabla S(x,t))^2 + U(x) = 0 \tag{2}$$

where $S(x,t)$ is Hamilton's principal function. The canonical equations of motion are recovered from (2) by means of its characteristics [1].

In any case, equation (2) can be derived from the basic notion of trajectory, either by means of extremals of Hamilton's principle [2], or, more directly, by an action principle formulated according to the deterministic control theory [3, 4]. Equation (2) has the role of the dynamic programming equation for a deterministic optimal control problem [5]. However, as will be discussed, it is possible to derive equation (2) directly, through a variational approach and an appropriate Lagrangian, without making any use of the notion of trajectory. The trajectories emerge as a by-product of this alternative derivation. This possibility is not so much significant from a technical point of view, as from a structural point of view. It allows us to place classical mechanics in a more flexible scheme.

Let us apply the operator ∇ to equation (2) [3, 4, 6]. We obtain

$$\partial_t v(x,t) + (v(x,t)\cdot\nabla)v(x,t) + \frac{1}{m}\nabla U(x) = 0 \qquad (v(x,t)\in\mathbb{R}^n) \tag{3}$$

with

$$v(x,t) = \frac{1}{m}\nabla S(x,t). \tag{4}$$

From a formal point of view equation (3) is nothing other than the familiar Euler equation for an ideal fluid in which the pression field is absent. Furthermore, equation (4) tells us that the flow of the fluid is irrotational. Now, if we consider for a fluid the Eulerian variables $\mu(x,t)$ (the mass density) and the velocity field $v(x,t)$, which are linked by the continuity equation

$$\partial_t \mu(x,t) + \nabla\cdot(\mu(x,t)v(x,t)) = 0 \tag{5}$$

it has been shown [7] that the Euler equation for irrotational flow follows from an action principle involving the Eulerian variables, a 'canonical' Lagrangian density and a constraint, which is given by equation (5). This result leads us to derive classical dynamics from a variational scheme which is particularly suitable for some extensions of the classical framework. This scheme will be considered in the following section.

2. Classical dynamics

First of all we consider the particle position a random variable and introduce its probability density $\rho(x,t)$. Instead of the trajectories $x(t)$, we plan to determine variationally $\rho(x,t)$. We do not invoke *a priori* some kind of stochastic process having $\rho(x,t)$ as its first-order distribution. The deterministic aspect of the problem which we are dealing, should emerge as a consequence of the equations which will determine $\rho(x,t)$.

Quite generally $\rho(x,t)$ has to satisfy a balance equation. We can write this equation in the form of a local conservation law

$$\partial_t \rho(x,t) + \nabla\cdot j(x,t) = 0 \tag{6}$$

by introducing a probability current density $j(x,t)$ ($j(x,t)\in\mathbb{R}^n$). We assume that $\rho(x,t)$ and $j(x,t)$ are smooth functions of x and t. Now, in terms of the field variables $\rho(x,t)$, $j(x,t)$ we introduce a Lagrangian density. For our model, by noting that j/ρ has the dimension of a velocity, we consider the 'canonical' Lagrangian density

$$\mathcal{L}(x,t) = \frac{m}{2}\left(\frac{j}{\rho}\right)^2 - U(x) \tag{7}$$

defined for every (x,t) where $\rho(x,t)\neq 0$.

Let us introduce a time interval $[t_0,t_1]$ and a compact domain Ω of $\mathbb{R}^n$, such that $\rho(x,t)>0$ for every $(x,t)\in\Omega\times[t_0,t_1]$. Then we build up an action integral, by averaging in Ω the density (7) with $\rho(x,t)$ and, furthermore, by taking into account that $\rho(x,t)$ and

$j(x,t)$ are not independent, but linked by constraint (6). With regard to that we introduce a Lagrangian multiplier field $\lambda(x,t)$ and assume that $\lambda(x,t)$ is also a smooth function. So we are led to consider the action integral

$$A[\rho,\, j,\, \lambda] = \int_{t_0}^{t_1} \mathrm{d}t \int_{\Omega} \mathrm{d}x \,[\mathcal{L}(x,t)\rho(x,t) + \lambda(x,t)(\partial_t \rho(x,t) + \nabla \cdot j(x,t))]\,. \tag{8}$$

Then the determining equations of the functions $\lambda(x,t)$, $j(x,t)$ and $\rho(x,t)$ can be obtained in every $(\Omega - \partial\Omega)(t_0,t_1)$, by requiring that $A[\rho,\, j,\, \lambda]$ is stationary under smooth independent variations of λ, j and ρ, having compact support contained in $(\Omega - \partial\Omega)(t_0,t_1)$. From the variation of $\lambda(x,t)$, we deduce immediately equation (6); so we fix our attention on ρ and j. Let us introduce smooth functions $\rho_0(x,t)$ and $j_0(x,t)(j_0 \in \mathbb{R}^n)$ with compact support in $(\Omega - \partial\Omega) \times (t_0,t_1)$ and define for $\epsilon \in \mathbb{R}$

$$\begin{aligned} \rho'(x,t) &= \rho(x,t) + \epsilon\rho_0(x,t) \\ j'(x,t) &= j(x,t) + \epsilon j_0(x,t). \end{aligned} \tag{9}$$

To first order in ϵ, after integration by parts, we obtain

$$\begin{aligned} &A[\rho',\, j',\, \lambda] - A[\rho,\, j,\, \lambda] \\ &\qquad = \epsilon \int_{t_0}^{t_1} \mathrm{d}t \int_{\Omega} \mathrm{d}x \left[\left(-\frac{1}{2} m \left(\frac{j}{\rho} \right)^2 - U(x) - \partial_t \lambda \right) \rho_0 + \left(m \frac{j}{\rho} - \nabla\lambda \right) \cdot j_0 \right]. \end{aligned} \tag{10}$$

Then, from the arbitrariness of ρ_o, j_0, we deduce

$$\frac{j}{\rho} = \frac{1}{m} \nabla\lambda \tag{11}$$

$$\partial_t \lambda + \frac{1}{2} m \left(\frac{j}{\rho} \right)^2 + U(x) = 0 \tag{12}$$

in every $(\Omega - \partial\Omega) \times (t_0,t_1)$.

We conclude that the determining equations of our functions are given by

$$\partial_t \lambda(x,t) + \frac{1}{2m} (\nabla\lambda(x,t))^2 + U(x) = 0 \tag{13}$$

$$\partial_t \rho(x,t) + \frac{1}{m} \nabla \cdot (\rho(x,t) \nabla\lambda(x,t)) = 0. \tag{14}$$

Equation (13) is nothing other than the Hamilton–Jakobi PDE (2). So the Hamilton principal function $S(x,t)$ can be identified with the Lagrangian multiplier field $\lambda(x,t)$. The crucial continuity equation (14) is a consequence of (6), (7) and (11). It is useful to point out that equation (11) is strictly related to the fact that we average $\mathcal{L}(x,t)$ with $\rho(x,t)$.

A basic feature of the above equations is that (13) is decoupled from (14) [8, 6]. As a consequence, once we have solved (13), we have a well-defined first-order PDE for $\rho(x,t)$ (equation (14)). This implies that (14) determines a deterministic motion and leads to the notion of trajectories [9].

Another important feature of the system of equations (13) and (14) is its invariance under the time-reversal transformations [8]:

$$\begin{aligned} &t \to t' = -t && x \to x' = x \\ &\rho(x,t) \to \rho'(x,t') = \rho(x,t) && \lambda(x,t) \to \lambda'(x,t') = -\lambda(x,t). \end{aligned} \tag{15}$$

In the previous considerations, classical dynamics has been rederived by a variational procedure which starts from a description in terms of a probability density $\rho(x, t)$ and makes use of two ingredients, a Lagrangian density and a balance equation (a local conservation law). Having two ingredients, the procedure is more flexible in the analysis of evolution phenomena from a variational point of view and suitable for taking into account some dynamical mechanisms which are not considered or neglected in the classical approach. As a matter of fact, we can have a dialectical relationship between the Lagrangian density and the balance equation, giving us the possibility of considering the concomitant action of an antagonistic process, which could lead to unwanted or meaningless effects, if taken separately.

3. Diffusions

Let us come back to our model of a particle of mass m in an external field of forces described by the potential $U(x)$. We will show how the previous variational scheme allows us to argue a correction or extension of the classical description of a motion.

First of all, having a description based on $\rho(x, t)$, we do not neglect the possibility that, even at the fundamental (or microscopic) level, can be operating a diffusion mechanism, of type familiar in the macroscopic world, expressing for example a general tendency to equalization. Of course, we have to solve the problem of how to reconcile this possibility with the absence, at the fundamental level, of any evident display of the effects of a diffusion process, such as the breaking of time-reversal invariance.

Formally, we suppose that the current density of the balance equation contains, besides a 'convection' (or 'drift' or 'transport') component (which we still call $j(x, t)$), also a 'diffusion' (or 'conduction') component $j_d(x, t)$, which depends in a given way on $\rho(x, t)$. In this paper, we limit ourselves to currents j_d which do not depend explicitly on (x, t) and are smooth local functions of $\rho(x, t)$. So we write

$$j_d(x, t) = j_d(\rho(x, t), \partial_i \rho(x, t), \partial_i \partial_j \rho(x, t), \cdots). \tag{16}$$

We note that $j_d(x, t)$, being a vector field, depends in an essential way on $\partial_i \rho(x, t)$. The local conservation law becomes

$$\partial_t \rho(x, t) + \nabla \cdot j(x, t) + \nabla \cdot j_d(\rho(x, t), \partial_i \rho(x, t), \partial_i \partial_j \rho(x, t), \cdots) = 0. \tag{17}$$

In our variational approach, the presence of a diffusion term in equation (17) has the consequence that also the Hamilton–Jacobi equation will be corrected through additional terms of dissipative or 'viscosity' type, if we keep the classical Lagrangian density of equation (7). Let us consider the action integral

$$A_d[\rho, j, \lambda] = \int_{t_0}^{t_1} \mathrm{d}t \int_\Omega \mathrm{d}x [\mathcal{L}(x, t)\rho(x, t) + \lambda(x, t)(\partial_t \rho(x, t) + \nabla \cdot j(x, t) + \nabla \cdot j_d(x, t))]. \tag{18}$$

The stationarity condition of A_d under the variation of j, again gives equation (11). However, from the stationarity of A_d under the variation of ρ we deduce, by taking into account equation (11), a modified Hamilton–Jacobi equation

$$\partial_t \lambda(x, t) + \frac{1}{2m}(\nabla \lambda(x, t))^2 + U(x) + \nabla \lambda \cdot \frac{\partial j_d}{\partial \rho}(x, t) - \partial_i \left(\nabla \lambda \cdot \frac{\partial j_d}{\partial(\partial_i \rho)}(x, t) \right) + \partial_i \partial_j \left(\nabla \lambda \cdot \frac{\partial j_d}{\partial(\partial_i \partial_j \rho)}(x, t) \right) - \cdots = 0 \tag{19}$$

which goes with (equation (17))

$$\partial_t \rho(x,t) + \frac{1}{m}\nabla \cdot (\rho(x,t)\nabla\lambda(x,t)) + \nabla \cdot j_d(x,t) = 0. \tag{20}$$

The system of equations (19) and (20) is not invariant under the time reversal transformations (15), since j_d, being a given function of ρ (see, for example, equation (21)), does not change sign under these transformations. Furthermore, in general, equation (19) is not decoupled from equation (20). However, with regard to this last aspect, there is a case in which we can reproduce the previous classical situation. Actually, if we consider the simplest structure of j_d, involving only a constant D_c (diffusion coefficient)

$$j_d^c = -D_c \nabla \rho(x,t) \tag{21}$$

we obtain the equations

$$\partial_t \lambda(x,t) + \frac{1}{2m}(\nabla\lambda(x,t))^2 + U(x) + D_c \triangle \lambda(x,t) = 0 \tag{22}$$

$$\partial_t \rho(x,t) + \frac{1}{m}\nabla \cdot (\rho(x,t)\nabla\lambda(x,t)) - D_c \triangle \rho(x,t) = 0 \tag{23}$$

where $\triangle = \nabla^2$. Equation (22) is decoupled from equation (23). But we have now a Hamilton–Jacobi equation with an additional 'viscosity' term [1, 5], providing a drift coefficient in a Fokker–Planck equation, which replace the continuity equation (14). So, also in this case, we can recover as a by-product the notion of trajectories, provided we replace the deterministic classical trajectories, associated with equations (13), (14), by the trajectories of a stochastic process related to a Brownian motion. As a matter of fact, these trajectories are the basic ingredients of a stochastic calculus of variations problem, for which equation (22) is the dynamic programming equation [5].

At the fundamental level, the above structure (equations (22), (23)) gives rise to several problems, even if it represents a very simple direct extension of the classical scheme. First of all, it conflicts with a strict requirement of time-reversal invariance. As a matter of fact, we will show in the following that this invariance can be strictly satisfied through the consideration of an additional dissipative mechanism. However, if we accept a weaker requirement on time-reversal invariance at the fundamental level, that is, its breaking is as small as we want, then a simple and direct formulation of this requirement is provided by the system of equations (22) and (23). Then we can argue that D_c is a very small constant which, on the basis of its physical dimensions (action/mass) could be expressed in the form

$$D_c = \frac{\hbar}{2M_c} \tag{24}$$

where $\hbar$ is the Planck constant and M_c a proper large mass.

There is another problem in the system of equations (22) and (23). With $D_c > 0$, equation (23) is a Fokker–Planck equation in forward form; its solution is associated with an initial condition. On the other hand, equation (22) has a 'backward' (or antidiffusion) structure; it requires a final condition for the field $\lambda(x,t)$ which determines the actual motion of our system. We shift elsewhere a close examination of the interesting role that this aspect can have in the analysis of an evolution phenomenon.

4. Quantum dynamics

Now, our variational scheme allows us to consider another possible dissipative mechanism, taking part in the law of motion. This mechanism, which is still related to the existence of an

internal vector field, given by the gradient of $\rho(x,t)$, can be introduced through the Lagrangian density.

We keep the classical structure given by equation (7), adding however the following possible correction. Quite generally in the description of motion in terms of trajectories, we can consider in the potential energy terms linear in the velocity, on the basis of the requirement that the forces are dynamical variables [4]. In our model, these terms are absent in the classical Lagrangian, since we have no external vector potential. However, in the description based on $\rho(x,t)$, there is available, in principle, an internal vector potential field, by the virtue of the presence of the vector field $\nabla\rho(x,t)$. So we can consider the Lagrangian density

$$\mathcal{L}_q(x,t) = \frac{1}{2}m\left(\frac{j}{\rho}\right)^2 + \frac{j}{\rho}(x,t)\cdot a(x,t) - U(x) \tag{25}$$

($a(x,t) \in \mathbb{R}^n$) where the vector-field potential $a(x,t)$ depends on ρ. We assume here that $a(x,t)$ depends only on ρ and that $\rho(x,t)a(x,t)$ is a smooth local function of this quantity.

But the coupling in $\mathcal{L}_q(x,t)$ between the current $j(x,t)$ and the gradient of ρ leads, in principle, to dissipative effects.

In the following, we will consider both the corrections to the classical description, given in equations (17) and (25). We note that, in the variational approach, the effective manifestation in the actual equations of motion of the dissipative aspects covered by equations (17) and (25), depends on the stationarity condition. In appropriate circumstances, such a condition can forbid the explicit and complete realization of the above diffusive mechanisms, relegating them only to a virtual role.

So we consider the new action integral

$$A_q[\rho, j, \lambda] = \int_{t_0}^{t_1} \mathrm{d}t \int_\Omega \mathrm{d}x[\mathcal{L}_q(x,t)\rho(x,t) + \lambda(x,t)(\partial_t\rho(x,t) + \nabla\cdot j(x,t) + \nabla\cdot j_d(x,t))]. \tag{26}$$

The stationarity condition of A_q, under the variation of j, gives

$$\frac{j}{\rho} = \frac{1}{m}\nabla\lambda - \frac{1}{m}a(x,t) \tag{27}$$

in every $(\Omega - \partial\Omega)\times(t_0,t_1)$.

A formal role of the term in equation (25) linear in j is to shift the classical stationary point of j. This has a relevant consequence. Using equation (27), the conservation law (17) becomes

$$\partial_t\rho(x,t) + \frac{1}{m}\nabla\cdot(\rho(x,t)\nabla\lambda(x,t)) + \nabla\cdot\left(j_d(x,t) - \frac{1}{m}\rho(x,t)a(x,t)\right) = 0. \tag{28}$$

From the stationarity condition of A_q under the variation of ρ and equation (27) we deduce

$$\begin{aligned}\partial_t\lambda + \frac{1}{2m}(\nabla\lambda)^2 + U + \nabla\lambda\cdot\frac{\partial}{\partial\rho}\left(j_d - \frac{\rho a}{m}\right) - \partial_i\left(\nabla\lambda\cdot\frac{\partial}{\partial(\partial\rho_i)}\left(j_d - \frac{\rho a}{m)}\right)\right)\\ + \partial_i\partial_j\left(\nabla\lambda\cdot\frac{\partial}{\partial(\partial_i\partial_j\rho)}\left(j_d - \frac{\rho a}{m}\right)\right) - \cdots + \frac{a^2}{2m} + \frac{\rho a}{m}\cdot\frac{\partial a}{\partial\rho}\\ - \partial_i\left(\frac{\rho a}{m}\cdot\frac{\partial a}{\partial(\partial\rho_i)}\right) + \partial_i\partial_j\left(\frac{\rho a}{m}\cdot\frac{\partial a}{\partial(\partial_i\partial_j\rho)}\right) - \cdots = 0.\end{aligned} \tag{29}$$

We see that the dissipative terms in equations (28) and (29) disappear if

$$\frac{\rho(x,t)a(x,t)}{m} = j_d(x,t). \tag{30}$$

In the following, we will assume such a relation. Then we obtain the system of equations

$$\partial_t \rho(x,t) + \frac{1}{m}\nabla\cdot(\rho(x,t)\nabla\lambda(x,t)) = 0 \tag{31}$$

$$\partial_t \lambda(x,t) + \frac{1}{2m}(\nabla\lambda(x,t))^2 + U(x) + \frac{a^2}{2m} + \frac{\rho a}{m}\cdot\frac{\partial a}{\partial\rho} - \partial_i\left(\frac{\rho a}{m}\cdot\frac{\partial a}{\partial(\partial\rho_i)}\right) + \partial_i\partial_j\left(\frac{\rho a}{m}\cdot\frac{\partial a}{\partial(\partial_i\partial_j\rho)}\right) - \cdots = 0. \tag{32}$$

This system satisfies time-reversal invariance. Such invariance appears as the result of a dynamical equilibrium between two diffusive mechanisms of opposite sign. As a consequence, the modified Hamilton–Jakobi equation (32) is now not decoupled from the continuity equation (31).

Let us consider some particular important cases of system (31), (32). If we assume we have partial differential equations of order not higher than two and quasilinear [1] and, furthermore, that the diffusion mechanism described by $j_d(x,t)$ acts in an homogeneous and isotropic way, we are led to consider quite generally the structure

$$j_d^q(x,t) = -D(\rho(x,t))\nabla\rho(x,t) \tag{33}$$

with a diffusion coefficient which is a smooth function of ρ and does not depend explicitly on x and t. As a matter of fact, problems of diffusing substances, with a diffusion coefficient depending on the concentration, are of particular theoretical and practical interest [10]. It is useful to introduce, as is done in the derivation of the hydrodynamical form of the Schrödinger equation (the Madelung fluid) [6, 8, 11], the scalar field $r(x,t)$ through

$$\rho(x,t) = \mathrm{e}^{2r(x,t)} \tag{34}$$

at the points (x,t) where $\rho(x,t) > 0$.

Furthermore let us set

$$\hbar(\rho(x,t)) = 2mD(\rho(x,t)). \tag{35}$$

The function $\hbar(\rho)$ has the physical dimension of an action. Then, by making use of (30), (33)–(35), we can write the system of equations (31) and (32) as

$$\partial_t r(x,t) + \frac{1}{2m}\triangle\lambda(x,t) + \frac{1}{m}\nabla r(x,t)\cdot\nabla\lambda(x,t) = 0 \tag{36}$$

$$\partial_t \lambda(x,t) + \frac{1}{2m}(\nabla\lambda(x,t))^2 + U(x) - \frac{1}{2m}[\hbar^2(\rho)\nabla r\cdot\nabla r + \hbar(\rho)\nabla\cdot(\hbar(\rho)\nabla r)] = 0 \tag{37}$$

at the points (x,t) where $\rho(x,t) > 0$.

We call the system of equations (36) and (37) the generalized Madelung fluid. We obtain the Madelung fluid [6, 8, 11] if we assume that $\hbar(\rho)$ does not depend on ρ and is equal to the Planck constant

$$\hbar(\rho) = \hbar. \tag{38}$$

As is well known, this very important case leads to the Schrödinger equation. In fact, when (38) is satisfied, by considering the complex quantity

$$\psi(x,t) = \exp\left(r(x,t) + \mathrm{i}\frac{\lambda(x,t)}{\hbar}\right) \tag{39}$$

one can deduce that the system of the two real equations (36) and (37) is equivalent to

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\triangle\psi(x,t) + U(x)\psi(x,t). \tag{40}$$

We note that the law $\lambda'(x,t') = -\lambda(x,t)$ in equation (15) is equivalent to the well-known implementation of time-reversal invariance of equation (40) by complex conjugation.

If we refer to quantum mechanics, we can infer from the actual knowledge of quantum phenomena that a possible dependence on ρ of the diffusion coefficient $D(\rho)$ should be extremely small. As a matter of fact, a constant $D(\rho)$ induces an exact linear structure through the introduction of the quantity $\psi(x,t)$. However, a possible dependence on ρ of $D(\rho)$ could be very interesting and have important consequences at some new scale of the quantum world.

A dissipative quantum mechanics would result if equation (30) is not satisfied exactly. In this case we can write

$$j_d(x,t) = j_d^q(x,t) + j_d^\delta(x,t) \tag{41}$$

and take $\rho(x,t)a(x,t) = mj_d^q(x,t)$. If we limit ourselves to currents characterized by a constant diffusion coefficient, so that

$$j_d^q(x,t) = -\frac{\hbar}{2m}\nabla\rho(x,t) \tag{42}$$

$$j_d^\delta(x,t) = -D_\delta\nabla\rho(x,t) \tag{43}$$

we obtain from (28) and (29) the equations of a dissipative quantum mechanics

$$\partial_t r(x,t) + \frac{1}{2m}\triangle\lambda(x,t) + \frac{1}{m}\nabla r(x,t)\cdot\nabla\lambda(x,t) - D_\delta(2\nabla r\cdot\nabla r + \triangle r) = 0 \tag{44}$$

$$\partial_t\lambda(x,t) + \frac{1}{2m}(\nabla\lambda(x,t))^2 + U(x) - \frac{\hbar^2}{2m}[\nabla r\cdot\nabla r + \triangle r] + D_\delta\triangle\lambda(x,t) = 0. \tag{45}$$

5. Discussions and outlook

We note that a structure like that of equation (32) has been postulated in the approach proposed in [6], related to the de Broglie–Bohm causal interpretation of quantum mechanics. In this approach, a 'generalized Hamilton–Jacobi theory' is advocated with the aim of overcoming the classical Lagrangian description and obtaining in this way a natural generalization of classical mechanics, providing the proper framework of the quantum theory. In this formulation, the Schrödinger equation needs a further postulate. We have shown in this paper that the spirit of 'natural' generalization of classical mechanics can be realized in the framework of a proper Lagrangian approach, which allows us also to deduce the Schrödinger equation.

The Madelung fluid has been derived by Nelson's stochastic mechanics approach [4, 12], where the basic ingredients are Brownian trajectories. A more convincing derivation has been obtained by Guerra and Morato [3] through a stochastic variational principle, but it requires Nelson's renormalization [4] of the action integral. It is not clear what really remains, after the renormalization, of the original Brownian trajectories. In this paper, we do not consider the trajectories as a primary notion; after all also in classical mechanics, as we have shown, they are a derived notion. In this respect, we are more akin to the spirit of quantum theory. But we follow the spirit of Nelson's stochastic mechanics when we introduce classical dissipative mechanisms for the description of quantum phenomena. We think that in this way stochastic mechanics becomes more flexible and can be a powerful tool, which goes

beyond a mere simulation of quantum mechanics and gives the means to explore new fields or to gain new insight in some relevant theoretical problems. In particular, we plan to extend the reformulation of the Hamilton principle for the motion of a particle, considered in section 2, to classical fields and to start from this extension for an analysis of the quantum theory of fields. We will consider also, within our approach, the quantum motion on a Riemannian manifold and, in this context, the possibility of giving an answer to the old problems in quantum mechanics related to the ordering of non-commuting operators; some analysis has already been done in the framework of Nelson's stochastic mechanics [13]. Furthermore, we hope that our consideration of the time-reversal invariance as the result of a dynamical equilibrium could be a useful starting point for an analysis of its breaking in the thermodynamic limit.

As is well known, in the remarkable Feynman path integral formulation [14] of quantum mechanics, the standard classical action obtained through $L(x(t), \dot{x}(t))$ of equation (1) has an important role. But this role is related to the type of approach, which presupposes non-classical concepts such as probability amplitudes and rules for them (so that the paths cannot be considered as actual possible paths of the particle). In the Feynman approach, the quantum dynamics is contained both in the standard classical action and in the rules of handling probability amplitudes. Our approach can be considered complementary: if we do not introduce at the start amplitudes and their rules, quantum dynamics can be deduced from a unique quantity, the action integral of equation (26), different from the standard classical action, in which classical dissipative mechanisms are taken into account.

There is another approach to quantum dynamics, which makes use of classical concepts: this is the Wigner formulation [15], in which the classical Liouville equation in the phase space is generalized to a quantum Liouville equation. In this case also classical concepts are accompanied by non-classical ones, such as pseudoprobabilities and rules which restrict the solutions of the evolution equation [15].

The above approaches, besides their theoretical relevance, are also very useful from a practical point of view, since they are the starting point of several numerical methods of simulations of quantum phenomena. From this point of view, we think that our approach, being completely variational, can make use of numerical techniques of the calculus of variations. These aspects will be investigated elsewhere.

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